Set Theory: Bridging Mathematics and Philosophy

Abstracts

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Contents

A Predicativisit Perspective on Definable Sets of Reals

30 Jul
11:15
Invited

Marianna Antonutti Mafori
Ludwig Maximilian University Munich

Definability hierarchies became an established area of study since the seminal work of Post and Kleene in the 1940s. Meaningful examples of definability hierarchies are taken to classify objects, or collections thereof, according to their definitional complexity; they are well-founded, increase in complexity at each level, and at each level new elements enter the hierarchy. In this talk, I will consider how certain definability hierarchies, those that now constitute descriptive set theory, arose from the work of the French and Russian analysts, focusing in particular on some concerns about definability and existence that emerge in the writings and correspondence of Borel, Baire, and Lebesgue. The formal techniques developed in the first half of the twentieth century provided a rich methodology and a precise framework to investigate properties of definable sets of reals such as what Kreisel [1960] calls "definitional completeness": if the hyper-arithmetical sets are constructed in a certain way, we eventually reach a stage where by iterating the construction operation, no new sets enter the hierarchy at that stage, nor there is an increase in definitional complexity. In the second part of the talk, I will discuss the importance of these results for the predicativist programme and reconstruct the view of predicativity outlined in Kreisel [1960]. In this perspective, it is possible to spell out how definable sets of reals can be considered as well-determined from a certain predicativist perspective.

An inconsistent multiverse?

30 Jul
16:00
Contributed

Carolin Antos and Daniel Kuby
Universität Konstanz

Recent studies in the philosophy of science have focused on inconsistent scientific theories and how scientists tolerate such inconsistencies. In this talk we want to analyze the situation of set-theoretic models in a multiverse picture, which we argue can be considered as an "inconsistency between models" case, such as, for example, models of ZFC + CH and ZFC + ¬CH. We will examine if different multiverse conceptions give rise to such an inconsistent picture and how inconsistency toleration by set-theoretic practitioners looks (and could look) like from these perspectives.

Philosophical implications of some recent breakthroughs in set theory

31 Jul
16:00
Invited

Joan Bagaria
Universitat de Barcelona

We shall discuss the foundational and philosophical implications of some recent results in set theory, such as Woodin's HOD-Dichotomy theorem, the proof by Aspero-Schindler that MM++ implies the (*) axiom, and some theorems, due to several authors, that provide new insights into the hierarchy of large cardinals, including large cardinals that contradict the Axiom of Choice.

The generic multiverse is not going away

31 Jul
12:05
Contributed

Douglas Blue
Harvard University

Views on the importance of forcing extensions to the question of whether there is an intended interpretation of set theory can be articulated with reference to the generic multiverse. Steel describes three such views. The *weak relativist* view is that all statements in the language of set theory are expressible in the language of the multiverse. The generic multiverse is all there is to set theory, in essentially the way that "generic multiversism" holds that the truths of set theory are those which are invariant in the generic multiverse. The *strong absolutist* view holds that there is an intended universe V of set theory, but the generic multiverse has no bearing on it. The generic multiverse language is too impoverished to capture the meaning of talk about V. The *weak absolutist* holds that there is a unique, definable world in the multiverse. It turns out that this is equivalent to the multiverse containing a model from which all others arise as forcing extensions, a "core."

Steel concludes that in light of the independence phenomena, unless and until we are in a position to specify V, it makes sense, for the weak relativist and weak absolutist at least, to conceive of set theory as taking place in the generic multiverse, rather than in an intended model.

It is now a theorem of Usuba that if there exists an extendible cardinal, then the weak absolutist view is true. Usuba's theorem can be interpreted as grounds for doing away with the generic multiverse. After all, the generic multiverse is a formalization of a conception of set theory (i) which was well supported by the independence phenomena and (ii) in which no universe of set theory, on the face of it, stands out over any other—that is, in which no V is specified initially. For the pluralist had a point: not enough had been done to specify the V which non-pluralists insisted exists. Whether there is such a V should be treated as an open question, and Steel put forth the generic multiverse as a framework for developing set theory in which an investigation into this question could take place.

Usuba's theorem shows that within this neutral framework, one *can* uniquely identify a world. The weak absolutist is vindicated. With Usuba's theorem in hand, the framework that allowed the core to be identified can be kicked away.

We will argue against this interpretation of Usuba's theorem by illustrating the utility of "hopping around in the multiverse" for acquiring mathematical knowledge. More precisely, we aim to establish that what we call "generic multiverse proofs" show that the generic multiverse provides problem solving tools for proving ZFC theorems, and insofar as it is useful in this way, it is here to stay as an object of foundational relevance.

Concepts and analogy-making in set-theoretic and category-theoretic foundations of mathematics

29 Jul
12:05
Contributed

Roland Bolz
Humboldt Universität zu Berlin

In this talk I investigate once more what is at stake in the category-theoretic interventions of Lawvere and others into the classical foundational discipline of set theory (Lawvere & Rosebrugh 2003, Lawvere & Schanuel 2009). The central claim of the paper is that category theorists take a qualitatively different approach to conceptual work in the foundations of mathematics than the prevalent logicist-inspired, post-Fregean tradition of set theory, complementing it in important ways by exhibiting *systematic analogies*. In my presentation I aim to describe this approach with more attention to talk about *concepts* and *conceptual labor* than has been done in the literature. Instead of assuming such talk to be entirely understood, I aim to develop a nuanced perspective on it by way of this example. In focusing on analogy-making I make systematic connections to recent work in cognitive linguistics concerning the *intimate link between concept formation and analogy-making* (Lakoff & Nunez 2000, Hofstadter & Sander 2013). I claim that two implicit doctrines in classical foundations have been an obstacle to the proper appraisal of the depth of the conceptual contributions of Lawvere and others to set theory: 1) a (relative) disconnection between questions of concept formation and questions of *understanding* and *insight*, and 2) the desideratum that foundational logical systems (ZF-languages are the chief example) must follow the syntax of natural language (the $\in$-sign is taken to capture one sense of the copula). Lawvere's category-theoretic contribution to foundations seems to do away with both doctrines by introducing a conceptual presentation of set-theoretic ideas which from the beginning shows *systematic analogies* to both simpler and more advanced mathematical concepts, using the ternary notion of functional composition as primitive. By focusing on the role of analogy-making in foundational studies, I aim to give more content to what Penelope Maddy has identified as category theorists' aim for **Essential Guidance** in the foundations of mathematics (Maddy, 2017). The final part of the talk concerns the concept of *concept* and *concept change* more generally, and distinguishes between a more narrow (and formal) *theory-internal* concept of concept and a wider concept of concept which is open to questions of cognition, salience, and familiarity.

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The V-logic multiverse

31 Jul
11:15
Contributed

Matteo de Ceglie and Claudio Ternullo
Ludwig Maximilian University Munich

In recent years, the notion of 'set-theoretic multiverse' has emerged and progressively gained prominence in the debate on the foundations of set theory. Several conceptions of the set-theoretic multiverse have been presented so far, all of which have advantages and disadvantages. Hamkins' *broad multiverse* ([Hamkins, 2012]), consisting of *all* models of *all* collections of set-theoretic axioms, is philosophically robust, but mathematically unattractive, as it may fail to fulfil fundamental foundational requirements of set theory. Steel's *set-generic multiverse* ([Steel, 2014]) consisting of all Boolean-valued models $V^{\mathbb{B}}$ of the axioms ZFC+Large Cardinals, is mathematically very attractive and fertile, but too restrictive. In particular, it cannot capture *all* possible outer models, focusing only on the set-generic extensions. Finally, Sy Friedman's *hyperuniverse conception* ([Arrigoni and Friedman, 2013]), although mathematically versatile and foundationally attractive, has the main disadvantage of postulating that V is countable.

In this paper, we introduce a new conception of the set-theoretic multiverse, that is, the 'V-logic multiverse', which expands on mathematical work conducted within the Hyperunuverse Programme ([Antos et al, 2015], [Friedman, 2016]), but also draws on features of the set-generic multiverse, in particular, on Steel's proposed *axiomatisation* of it.

V-logic is an *infinitary* logic (a logic admitting formulas and proofs of infinite length) whose language $\mathcal{L}_{\kappa^+,\omega}$, in addition to symbols already used in first-order logic, consists of κ-many constants $\overline{a}$, one for each set $a \in V$, and of a special constant symbol $\overline{V}$, which denotes V. In V-logic, one can ensure that the statement asserting the consistency of ZFC+ψ, for some set-theoretic statement ψ, is satisfied by some model M, *if and only if* M is an outer model of V. By outer model we mean here: models obtained through *set-forcing, class-forcing, hyperclass-forcing* and, in general, any model-theoretic technique able to produce *width extensions* of V. Thus, through the choice of suitable consistency statements, we can generate outer models M, endowed with specific features. The V-logic multiverse is precisely the collection of all such outer models of V.

The following observations help illustrate the adequacy of our method to produce a multiverse concept which, in our view, has better prospects than the ones mentioned above:

1. Contrary to the set-generic multiverse, the V-logic multiverse is broad enough to include all kinds of outer models.

2. Contrary to the hyperuniverse conception, the V-logic multiverse does not reduce to a collection of countable transitive models, as V does not need to be taken to be countable.

As it stands, the V-logic multiverse may be used to pursue two fundamental research directions, both of which are ideally aimed at developing an *axiomatic theory* of the multiverse.

One consists in defining the V-logic multiverse of different extensions of ZFC, by taking into account such axioms as AD, PD, large cardinals, $V = L$ and others, and investigating which relationships obtain among all such V-logic multiverses.

The second direction consists in taking V to be approximated by different structures, such as L, L-like models, V_κ, where κ is some large cardinal and investigate, for instance, whether members of the corresponding V-logic multiverses are compatible with each other, and to what extent. For instance, the L-logic multiverse maximises compatibility, but reduces the extent of structural variability among universes, thus reducing the range of alternative *truth outcomes* in the multiverse.

We argue that the V-logic multiverse is both mathematically more fruitful and philosophically robust than all the other multiverse conceptions, and consequently the best candidate to be the foundation of set theory and mathematics.

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Is set theory pure or applied mathematics? On the ontological power of set theory and its limits

30 Jul
10:00
Invited

Mirna Džamonja
University of East Anglia

The talk is inspired by the reading of Alain Badiou's book "Immanences des vérités". This book closes a circle of Badiou's engagement with the higher infinite, set theory, category theory, forcing and large cardinals, as part of his philosophical tool-case developed through his four large books. Badiou makes of set theory an applied science and applies the set-theoretic thinking to philosophical and political contexts. Forcing has a special place in this philosophy. Badiou's work is well-known and has admirers as much as adversaries. My intention has been to develop my own, hopefully balanced, opinion and to see if the relation between set theory and this philosophy can be productive for set theory in the sense of inspiring new ideas and directions of reserach. I will promote one such idea through "the index of absoluteness"."

Categories of amenable embeddings and what canonicity in set theory cannot be

30 Jul
14:20
Contributed

Monroe Eskew
Universität Wien

In recent work with Sy Friedman, we explored a notion of width reflection: $V \subseteq W$ are models of ZFC with the same ordinals, and there is an elementary embedding $j : V \to W$ such that j is an amenable predicate for W—equivalently, that $j[x] \in W$ for all $x \in V$. By Kunen's Theorem, $V \neq W$, unless the critical sequence is cofinal in the ordinals. Although this notion looks like a very strong large cardinal property, it can be obtained from less than a measurable cardinal. The main focus of the paper is to explore what kinds of structures can be found among systems of such embeddings.

Given an ordinal δ, let $\mathcal{E}_\delta$ be the category whose objects are all transitive models of ZFC of height δ and whose arrows are all elementary embeddings between these models. Let $\mathcal{A}_\delta$ be the subcategory where we take only amenable embeddings as arrows. (It is easy to see that amenable embeddings are closed under composition.) Partial orders are naturally represented as categories where between any two objects there is at most one arrow. Let us say that a subcategory $\mathcal{D}$ of a category C is *honest* if whenever x and y are objects of $\mathcal{D}$ and there is an arrow $f : x \longrightarrow y$ in C, then there is one in $\mathcal{D}$ as well.

> If there is a countable transitive model of ZFC of height δ satisfying that there is a proper class of large enough cardinals, then $\mathcal{A}_\delta$ contains honest subcategories isomorphic to:
>
> 1. The real numbers.
> 2. An Aronszajn tree.
> 3. A universal countable pseudotree.

We would like to discuss the implications of the above result for the possible meanings of "canonical model" in set theory. It shows that whatever axioms we adopt that are consistent with having many large cardinals, the resulting theory cannot dictate a canonical way to build the universe along the ordinals. Epistemically, it seems to say that if we believe in a strong enough theory of sets, then our universe may be just one among a rich multiverse of models of the same height that are indistinguishable by first-order properties. Canonicity fails very badly, since we have literally a continuum of possibilities. We might have built the universe with more or fewer sets, with very little change in the total information. If we move through this multiverse by expanding or contracting our current universe, then this process is not canonical either, since we can always do that a little more or a little less. Furthermore, the tree constructions show that we may move into severely incompatible universes, which are nonetheless indistinguishable from the internal point of view.

Boolean-valued sets as arbitrary objects

29 Jul
10:00
Invited

Leon Horsten

University of Bristol

This paper explores the connection between boolean-valued class models of set theory and the theory of arbitrary objects in Kit Fine's sense of the word. In particular, I explore the hypothesis that the set theoretic universe as a whole can be seen as an arbitrary entity, which can in turn be taken to consist of arbitrary objects (arbitrary sets).

A question about empirical investigations for philosophy

30 Jul
15:30
Contributed

Deborah Kant

Universität Konstanz

Philosophers of mathematics are more and more interested in mathematical pratices. It is, however, not completely understood how empirical investigations can inform philosophy. We systematise the interactions between the disciplines, and argue that social phenomena are properly investigated by empirical methods and that philosophy informs the theory and research questions of empirical investigations. We review three examples from the philosophy of mathematics and mathematical practices on their methodology, present the author's interview study with professional set theorists, and end with an open question: How do the social phenomena of the mathematical community relate to our philosophy of mathematics?

On Invariance in foundational practice

29 Jul
16:15
Invited

Juliette Kennedy

University of Helsinki

Set-theoretic truth

30 Jul
16:30
Invited

Godehard Link

Ludwig Maximilian University Munich

I start with giving a non-Platonist account of general mathematical truth that does justice to our strong anti-formalist intuitions without embracing what W. Tait calls a "superrealist" ontology of a Model-in-the-Sky as the eternal truth-maker of mathematical statements. "Non-Platonist" instead of "anti-Platonist" is meant to indicate that I intend to sidestep the opposition of Platonism vs nominalism. In fact, I argue that almost all revisionist stances in the philosophy of mathematics are rather unattractive, in particular formalism, intuitionism, and so-called fictionalism. Instead, the conception of mathematical truth I favor is highly theoretical, continuous with science, locally semantic (or interpretive), and globally coherentist. Since I am also unconcerned about the indispensability argument, I am in line with much of modern set-theoretical practice where the techniques of forcing and inner models play a dominant role. However, in the face of the plethora of models generated by these technical tools suggesting relativism of some sort, I still think that a substantive notion of set-theoretic truth can be defended along the lines proposed here. In this context I will give my take on Hugh Woodin's quest for the Ultimate-L.

Iterability and generalised proof theory

31 Jul
10:00
Invited

Toby Meadows

UC Irvine

This paper will present an approach to strengthening logic that considers a sequence of generalised Gödel sentences and addresses the underlying incompleteness using large cardinals and iteration. This account has philosophical applications, for example, with respect to Maddy's notion of a fair interpretation. Two different approaches to iterability (from inner model theory and proof theory) will be discussed and some speculative comparisons will be made.

Modal set theory and potential hierarchies

29 Jul
14:00
Contributed

Christopher Menzel and Guillermo Badia

Texas A&M University and University of Queensland

As is well known, Russell's Paradox motivated the development of axiomatic set theory, and its natural "model" — viz., the cumulative hierarchy of sets — arguably provided a satisfying structural explanation of where the reasoning in the paradox goes wrong. There is thus an undeniably robust foundation for *iterative set theoretic (IST) realism*, that is, realism about the cumulative hierarchy. Two fundamental intuitions lie at the heart of IST realism. The first, of course, is that sets are "constructed" in stages such that, beginning (perhaps) with an initial plurality of urelements, each stage consists of all the sets that can be formed from the things in the preceding stage. Call this the *iterative intuition*. The second — call it the *realist intuition* — is that all the sets are *there*, indeed, necessarily so in the case of pure sets. But these two fundamental intuitions of IST realism themselves give rise to a paradox of their own. For if all the sets are *there*, then why does the iterative intuition not apply to *them*? Why does the hierarchy not continue on, starting with the sets there in fact are, into yet higher levels? On the face of it, the two intuitions cannot both be true.

A lot of interesting and important work, notably by Charles Parsons and more recently by Øystein Linnebo, has brought modal logic and set theory together into a framework that, by spelling out the idea that the cumulative hierarchy is merely *potential*, promises (i) to reconcile the apparently conflicting intuitions that underlie IST realism, and moreover (ii) to account for the initially compelling intuitions underlying the principles that lead to Russell's paradox. However, in this paper, I will first argue that, while modal set theory solves the realist's dilemma if the modality is taken to be genuinely metaphysical, such *modal set theoretic (MST) realism* suffers from a similar and equally serious problem about the metaphysics of sets, viz., the apparent *modal capriciousness* of the existence of sets. The problems with both IST and MST realism cast doubts on the viability of either brand of realism. I will argue that the best hope for a reasonably robust mathematical realism about set theory lies with the modal structuralist program first proposed by Hilary Putnam and developed in great detail by Geoff Hellman. After discussing some questions that still remain about this program, I will examine an argument that the integrity of the view is preserved only by rejecting necessitism, i.e., the view (recently defended at extraordinary length by Timothy Williamson) that, necessarily, everything exists necessarily. I will close by examining possible rejoinder for the necessitist

based upon a revision of Linnebo's modal set theory that allows for the existence of absolutely infinite sets of bounded rank that suggests that, while absolute generality is possible with regard to the domain of necessary objects, our set theoretic discourse must always be restricted to some set-sized natural model V_κ for κ inaccessible.

Incomparable Extensions of ZF

29 Jul
14:50
Invited

Karl-Georg Niebergall
Humboldt-Universität zu Berlin

Through application of a lemma by Lindström we can prove the existence of theory extensions of PA, which are incomparable regarding consistency strength and relative interpretability. We will see how this construction can be transferred to the context of ZF.

Cantor's Paradise on Skolem's Earth

30 Jul
13:30
Contributed

Mangesh Patwardhan
National Insurance Academy Pune

As a matter of conceptual coherence, the study of set theory and its models cannot get off the ground unless we have a stock of sets that are already available to us; as set or class models of set theory are made up of, well, sets themselves or their collections. Therefore, it seems that not only universists but even radical multiversists like Hamkins have to reckon with the need for such stock of sets. In recent years, the iterative set theoretic hierarchy "V" has been widely accepted as the stock of all sets that are there, in some sense. However, it raises several issues. One, as Quine remarked, this conception seems to carry staggering ontological presuppositions. Two, there is the issue of width and height potentialism. Proponents of width potentialism such as Feferman maintain that the concept of powerset is vague, even at the "lowly" level of the continuum. The issue whether giving a second order characterization of set theory and thereby V involves an illicit and viciously circular appeal to the powerset concept itself continues to be debated. Even leaving that aside, the problem of height potentialism remains. There is no principled way to decide where one should stop iterating the powerset operation and take the collection of sets constructed till then as V (provided it is a strongly inaccessible rank). As Maddy remarks, it is difficult to say why the "powersets" atop V is not just another stage we forgot to include. In fact, Zermelo visualised the set theoretic universe as an unlimited progression of models of set theory with no true end, but only relative stopping points. The reformulation by Shepherdson (endorsed by Isaacson) of Zermelo's argument as proof in first order ZF regarding a class of full inner models of NBG is philosophically unsatisfactory. It can also be argued that the existence of large large cardinals does not follow from the iterative picture. Other sophisticated philosophical considerations such as uniformity, generalization and inexhaustibility have been brought in to justify these. I propose that we turn Skolem's criticisms of set theory and in particular "Skolem's paradox" on their head to get an initial stock of sets and get the model theory of set theory going. His "numerical model" version - if an axiom system A (such as ZFC) is consistent, it has a model in natural numbers – fits the bill. This approach should be acceptable to those troubled by the staggering ontology implicit in the iterative universe picture as well as by potentialism issues. Moreover, it should be unproblematic to Maddy's second philosopher. Skolem's contention that in light of his analysis, the theorems of set theory can be made to hold in a mere verbal sense actually becomes a virtue. I argue that this formulation allows us to enjoy the beauty of Cantor's paradise while remaining grounded on Skolem's earth.

Can all things be counted?

29 Jul
11:15
Contributed

Chris Scambler
New York University

Recent work in modal set theory has seen the development of formal theories encapsulating 'indefinite extensibility' responses to the Russell paradox (e.g. in the work of Linnebo and Studd). The guiding idea is that although the Russell paradox shows that (necessarily) there are some things that don't form a set, nevertheless any (possible) things possibly form a set. Such theories have Kripke models whose possible worlds are ranks of the cumulative hierarchy, and they can also be shown to interpret standard first order set theories under 'potentialist translations'. As a result many of the benefits that are offered by the combination of the standard iterative conception of sets and Zermelo-Fraenkel set theory are also offered by these theories, even though the latter do not posit the existence of any 'special' things that can't form a set.

This article offers a formal theory that, in addition to endorsing an indefinite extensibility solution to the Russell paradox, also endorses an indefinite extensibility 'solution' to Cantor's theorem. The guiding idea is that although Cantor's theorem shows that (necessarily) there are some things that are not in the range of a function defined on the natural numbers, nevertheless any possible things are possibly in the range of such a function. I give a consistency proof (relative to standard set theory) for a theory in modal and plural logic that formalizes these ideas by adapting the system of 'generic multiverse rank potentialism' studied by Linnebo and Hamkins. I then discuss potentialist translations of standard set theories into the theory. I show that although the developed theory interprets full standard set theory under a restricted interpretation, nevertheless the full modal language satisfies a natural translation of 'all sets are countable'. Just as the modal indefinite extensibility theories for the Russell put pressure on the idea that there are some special things that can't form a set, I suggest that the modal indefinite extensibility theory for Cantor's theorem puts pressure on the idea that there are some special things that can't be in the range of a function on the natural numbers. I suggest that this shows that uncountable infinities are not really needed, even for standard set theory.

The singularity of forcing

31 Jul
14:00
Contributed

Thomas Tulinski

École Normale Supérieure de Lyon

The *significance* of set forcing for the philosophy of set theory follows from this that it provides a powerful tool to prove independence results (problem of pluralism and of absolutely undecidable statements), to show the underdetermination of transfinite cardinal arithmetic (e.g., Easton's theorem) and, more generally, the universe of sets (problem of the categoricity of the reference). But also from this that it provides with new axioms set theory (forcing axioms), a new conception of set-theoretical truth (the multiverse conception of truth), new set-theoretical methods (set-theoretical geology and the modal logic of forcing) and new concepts of set (set-theoretic multiverses).

However, the *singularity* of forcing (i.e., what characterizes it uniquely) remains questionable: from the start, there have been noticeable analogies between forcing and, respectively, Kleene intuitionistic realizability of formal statements and proofs in Heyting's arithmetic, and Kripke semantics for intuitionistic and modal logic. All the more since these two notions have been generalized in such a way that forcing appears as but a degenerate particular case of each.

On the one hand, the development of categorical logic, driven in particular by the desire of a category-theoretic formulation of the independence of CH and AC, has provided topos theory with what is known as *Beth-Joyal semantics*, generalizing indeed both Kripke semantics and forcing semantics. On the other hand, thanks to a discovery by Griffin allowing a computational interpretation of LEM, thereby contributing to Kreisel's *unwinding program*, Krivine has been able to define *classical realizability* so as to express the computational content of formal classical statements such as axioms of ZF and DC by producing ZF models that have a fundamentally different structure than that of forcing models.

The purpose of my talk is to show *that* one can gain worthy insights into set forcing by studying its relationships to the aforementioned notions, *how* can one hope to get such insights and *what* some of them are. In particular, I will show that analyzing forcing from the point of view of Beth-Joyal semantics and classical realizability is in some sense indispensable in order to account for two apparently paradoxical features of forcing.

First, it is paradoxical that forcing implicitly obeys intuitionistic rules even though it has been formulated in a context where all usual classical indicators were satisfied (i.e., law of the excluded middle, axiom of choice, Booleanness). Second, it is paradoxical that forcing has any computational content at all since it involves many highly non-constructive objects (e.g., uncountable models of ZF and generic sets).

As a result, a picture of *conceptual change* via forcing suggests itself. Not only will forcing appear as an instance of the dialectical process of the negation of constancy, by generating a topos which is provably a IZF model, i.e., the theory of variable sets, but the terminal constancy obtained by negating the previous negation, i.e., set forcing invariance, will be justified as classically realizable.

A semantic approach to independence

31 Jul
14:50
Contributed

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A practical response to the philosophical issues that followed the development of mathematics at the end of the XIX century consisted in a revival of the axiomatic method. Through the innovative work and the authority of David Hilbert, the simultaneous development of logic and mathematics lead to an attempt to discharge the ontological and epistemological controversies onto the axiomatic presentation of a theory. Even set theory underwent the same transformation and eventually ZFC was accepted as a foundation for the whole mathematics. Nonetheless the faith in the axiomatic solution to every set-theoretical problem was proved unfounded by the plethora of independent results produced by forcing. In order to overcome the problem that the independence phenomenon posed to mathematical truth, Gödel [3] proposed to extend ZFC with new principles able to solve questions like CH. Contrary to Gödel's hope in an uncontroversial axiomatic solution, his program opened once again the philosophical debates that the axiomatic method was meant to solve. Indeed, the so-called Gödel's program has shown its limits in deciding between competing, incompatible extensions of ZFC [5], [1]. But, then, how to overcome the limits of independence by purely mathematical tools?

In order to approach this problem we suggest to turn upside down the traditional perspective on independence and instead of completing the syntactic side, we propose to complete set theory semantically by applying to set theory model theoretic techniques meant to produce complete models. The study of model-theoretics tools to complete theories goes back to the work of Robinson and to the concept of model completeness from the period 1950–1957. This notion, together with those of existentially complete theories and model companionships, aimed at generalizing to other algebraic contexts, the peculiar role that algebraically closed fields play with respect to the class of fields.

We will discuss the philosophical significance of this new approach together with some preliminary results obtained by the application of Robinson infinite forcing to the collection of set-forcing extensions of models of ZFC.

In [4] it is shown that Robinson infinitely generic structures exist and that they validate many generic absoluteness principles like Maximality Principles, Resurrection Axioms, and Bounded Forcing Axioms. This unified perspective, thus, shows an interesting connection between a semantic perspective and generic absoluteness results. Moreover, in [5], this new model theoretic approach and Woodin's classical results on the absoluteness of second order arithmetic are combined to show the existence of the model companion of ZFC plus large cardinals.

We believe that we are only scratching the surface of a new promising interaction between model theory and set theory, able to produce interesting new results and a deeper understanding of set theory.

References

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